Application of KKT Conditions

For a problem in the following form,

Min	$f(\mathbf{x})$	(1)
s.t.	$g_i(\mathbf{x}) - b_i \geq 0 \quad i = 1, \ldots, k$	(2)
	$g_i(\mathbf{x}) - b_i = 0 \quad i = k+1, \ldots, m$	(3)

A) Give the KT necessary conditions, explaining each equation.

Equation	Explanation
$g_i\left(\mathbf{x}^*\right) - b_i$ is feasible $\quad i = 1, \ldots, m$	Primal Feasibility
$\nabla f\left(\mathbf{x}^*\right) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i\left(\mathbf{x}^*\right) = \mathbf{0}$	Dual Feasibility No direction which improves objective and is feasible
$\lambda_i^*\left[g_i\left(\mathbf{x}^*\right) - b_i\right] = 0 \quad i = 1, \ldots, k$	Complementary slackness
$\lambda_i^* \geq 0 \quad i = 1, \ldots, k$	Positive Lagrange multipliers

B) A cylindrical storage tank is to be constructed for which the following costs apply:

Metal for sides	\$30.00/sq. ft.
Concrete base and metal bottom	\$37.50/sq. ft.
Top	\$7.50/sq. ft.

The tank is to be constructed with dimensions such that the cost is a minimum for whatever capacity is selected. One possible approach to selecting the capacity is to build the tank such that an additional cubic foot of capacity costs \$8. (Note this does not mean \$8 per cubic foot average for the entire tank.) Find the optimal diameter and height of the tank.

$\text{dollars} := 1$

$c_{side} := 30 \cdot \frac{\text{dollars}}{\text{ft}^2} \qquad c_{base} := 37.50 \cdot \frac{\text{dollars}}{\text{ft}^2} \qquad c_{top} := 7.50 \cdot \frac{\text{dollars}}{\text{ft}^2}$

$d = \text{Diameter} \qquad h = \text{height} \qquad V = \text{capacity}$

$A_{side} = \pi \cdot d \cdot h \qquad A_{base} = A_{top} = \frac{\pi}{4} \cdot d^2$

min total cost

s.t. Volume Equation

min $c_{total} = c_{side} \cdot A_{side} + (c_{top} + c_{base.}) \cdot A_{base} = c_{side} \cdot (\pi \cdot d \cdot h) + (c_{top} + c_{base.}) \cdot \left(\frac{\pi}{4} \cdot d^2\right)$

s.t. $V = \frac{\pi}{4} \cdot d^2 \cdot h$ $\lambda_1 := 8 \cdot \frac{dollars}{ft^3}$

Write the KT Conditions

1. Primal Feasibility

$$\frac{\pi}{4} \cdot d^2 \cdot h - V = 0$$

2. Dual Feasibility

$$\begin{bmatrix} c_{side} \cdot \pi \cdot h + (c_{top} + c_{base.}) \cdot \frac{\pi}{2} \cdot d \\ c_{side} \cdot \pi \cdot d \end{bmatrix} - \lambda_1 \cdot \begin{pmatrix} \frac{\pi}{2} \cdot d \cdot h \\ \frac{\pi}{4} \cdot d^2 \end{pmatrix} = 0$$

3. Complementarity slackness and 4. Positive Lagrange multipliers don't apply

Solve KT Conditions

Start with last equation in dual feasibility section

$c_{side} \cdot \pi \cdot d - \lambda_1 \cdot \frac{\pi}{4} \cdot d^2 = 0$ Eliminate d=0 solution $c_{side} \cdot \pi - \lambda_1 \cdot \frac{\pi}{4} \cdot d = 0$

Solve for d $d := \frac{4c_{side}}{\lambda_1}$

$d = 15 \cdot ft$

Use first dual feasibility equation to solve for h

$$c_{side} \cdot \pi \cdot h + (c_{top} + c_{base.}) \cdot \frac{\pi}{2} \cdot d - \lambda_1 \cdot \frac{\pi}{2} \cdot d \cdot h = 0$$

$$h = \frac{-(c_{top} + c_{base.}) \cdot \frac{\pi}{2} \cdot d}{c_{side} \cdot \pi - \lambda_1 \cdot \frac{\pi}{2} \cdot d} = \frac{-(c_{top} + c_{base.}) \cdot d}{2c_{side} - \lambda_1 \cdot d}$$ $$h := \frac{-(c_{top} + c_{base}) \cdot d}{2c_{side} - \lambda_1 \cdot d}$$

$h = 11.25ft$

Use Primal Feasibility Constraint to solve for V

$V := \frac{\pi}{4} \cdot d^2 \cdot h$ $V = 1988 \cdot ft^3$