Karush-Kuhn-Tucker Conditions
with Inequality and Equality Constraints

For a problem in the following form,

Min	$f(\mathbf{x})$	(1)
s.t.	$g_i(\mathbf{x}) - b_i \geq 0 \quad i = 1, \ldots, k$	(2)
	$g_i(\mathbf{x}) - b_i = 0 \quad i = k+1, \ldots, m$	(3)

A) Give below the KT necessary conditions, explaining each equation.

Description	Equation	Applies to
Feasibility	$g_i\left(\mathbf{x}^*\right) - b_i$ is feasible $\quad i = 1, \ldots, m$	2,3
No direction which improves objective and is feasible	$\nabla f\left(\mathbf{x}^*\right) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i\left(\mathbf{x}^*\right) = \mathbf{0}$	1-3 (all)
Complementary slackness	$\lambda_i^*\left[g_i\left(\mathbf{x}^*\right) - b_i\right] = 0 \quad i = 1, \ldots, k$	2
Positive Lagrange multipliers	$\lambda_i^* \geq 0 \quad i = 1, \ldots, k$	2

B) Solve for the optimum using the KKT conditions

Min $\quad f = 4x_1^2 + 2x_2^2$

s.t. $\quad 3x_1 + x_2 = 8$

$\quad 2x_1 + 4x_2 \leq 15$

Note: at the optimum, it is known that the inequality constraint is satisfied but not binding. Take advantage of this information.

<u>Solution</u>

$$3x_1 + x_2 - 8 = 0$$

$$-2x_1 - 4x_2 + 15 \geq 0$$

Not a binding constraint so $\lambda_2 = 0$ and we drop the inequality constraint.

$$\begin{bmatrix} 8x_1 \\ 6x_2 \end{bmatrix} - \lambda_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A := \begin{pmatrix} 3 & 1 & 0 \\ 8 & 0 & -3 \\ 0 & 6 & -1 \end{pmatrix} \quad b := \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix} \quad x := A^{-1} \cdot b \quad x = \begin{pmatrix} 2.323 \\ 1.032 \\ 6.194 \end{pmatrix}$$

C) For the following problem,

Min $\quad f = x_1^2 + x_2$

s.t. $\quad g_1 = x_1^2 + x_2^2 - 9 \le 0$

$\quad g_2 = x_1 + x_2 - 1 \le 0 \quad$ ++

Show that the point [1,0] does not satisfy the KKT conditions

1 – Check feasibility of all equations

$x_1 := 1 \qquad x_2 := 0$

$x_1^2 + x_2^2 - 9 = -8$ This is less than or equal to zero - passes test

$x_1 + x_2 - 1 = 0$ This is less than or equal to zero - passes test

Translate Inequality Constraints to Standard Form

$$g_1 = -x_1^2 - x_2^2 + 9 \ge 0$$

$$g_2 = -x_1 - x_2 + 1 \ge 0$$

Calculate Lagrange multipliers

$$\begin{pmatrix} 2 \cdot x_1 \\ 1 \end{pmatrix} - \lambda_1 \cdot \begin{pmatrix} -2 \cdot x_1 \\ -2 \cdot x_2 \end{pmatrix} - \lambda_2 \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 0$$

with $x_1 = 1$ and $x_2 = 0$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$\lambda_2 := -1 \qquad 2 \cdot \lambda_1 + \lambda_2 = -2 \qquad \lambda_1 = -\frac{1}{2}$

$\lambda_1 < 0$ and

$\lambda_2 < 0$ violate KKT condition #4

Set λ_1=0 and re-evaluate because contour plot shows that constraint #1 is not active:

However, when this is performed, λ_2=-2 and λ_2=-1 result from KKT condition #2 equations and a consistent set of Lagrange multipliers cannot be obtained. Therefore, it is not at the optimal solution.