Karush-Kuhn-Tucker Conditions for Equality Constraints

For a problem in the following form,

Min	$f(\mathbf{x})$	(1)
s.t.	$g_i(\mathbf{x}) - b_i \geq 0 \quad i = 1, \ldots, k$	(2)
	$g_i(\mathbf{x}) - b_i = 0 \quad i = k+1, \ldots, m$	(3)

A) Give below the KKT necessary conditions, explaining each equation.

Description	Equation	Applies to
Feasibility	$g_i\left(\mathbf{x}^*\right) - b_i$ is feasible $\quad i = 1, \ldots, m$	2,3
No direction which improves objective and is feasible	$\nabla f\left(\mathbf{x}^*\right) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i\left(\mathbf{x}^*\right) = \mathbf{0}$	1-3 (all)
Complementary slackness	$\lambda_i^*\left[g_i\left(\mathbf{x}^*\right) - b_i\right] = 0 \quad i = 1, \ldots, k$	2
Positive Lagrange multipliers	$\lambda_i^* \geq 0 \quad i = 1, \ldots, k$	2

B) Given the following problem, solve for the solution using the KKT Conditions

Min $\quad f = 2x_1^2 + x_2^2 + 4x_3^2$

s.t.

$$g_1 = x_1 + 2x_2 - x_3 = 6$$
$$g_2 = 2x_1 - 2x_2 + 3x_3 = 12$$

$$g_i\left(\mathbf{x}^*\right)-b_i \text{ is feasible} \quad i=1,\dots,m$$

$$x_1 + 2x_2 - x_3 - 6 = 0$$

$$2x_1 - 2x_2 + 3x_3 - 12 = 0$$

$$\nabla f\left(\mathbf{x}^*\right)-\sum_{i=1}^{m}\lambda_i^*\nabla g_i\left(\mathbf{x}^*\right)=\mathbf{0}$$

$$\begin{bmatrix} 4x_1 \\ 2x_2 \\ 8x_3 \end{bmatrix} - \lambda_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \lambda_2 \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = 0$$

5 Equations, 5 Unknowns

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 2 & -2 & 3 & 0 & 0 \\ 4 & 0 & 0 & -1 & -2 \\ 0 & 2 & 0 & -2 & 2 \\ 0 & 0 & 8 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A := \begin{pmatrix} 1 & 2 & -1 & 0 & 0 \\ 2 & -2 & 3 & 0 & 0 \\ 4 & 0 & 0 & -1 & -2 \\ 0 & 2 & 0 & -2 & 2 \\ 0 & 0 & 8 & 1 & -3 \end{pmatrix} \qquad b := \begin{pmatrix} 6 \\ 12 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad A\cdot x = b \qquad x := A^{-1}\cdot b \qquad x = \begin{pmatrix} 5.045 \\ 1.194 \\ 1.433 \\ 7.522 \\ 6.328 \end{pmatrix}$$