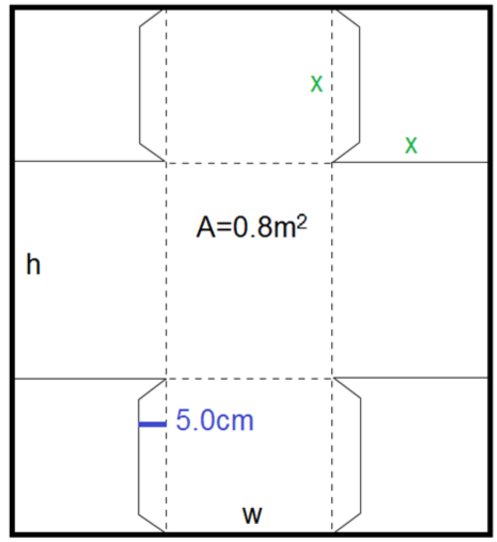
Optimization: Maximize the Volume of a Box

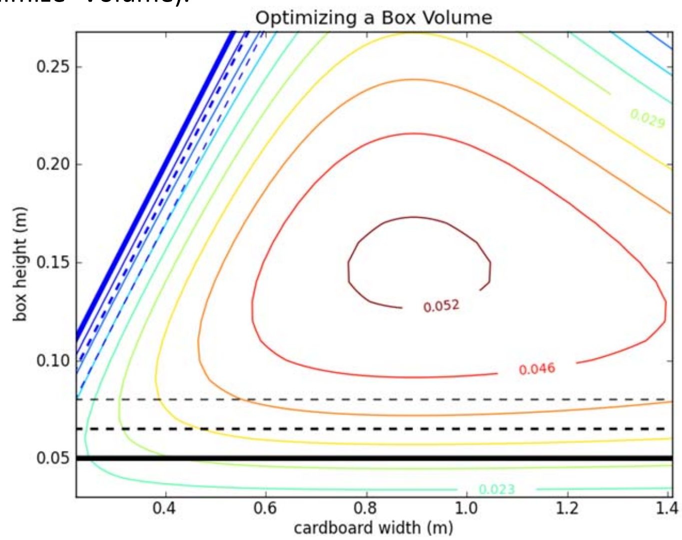


A piece of cardboard with a total area of $0.8m^2$ is to be made into an open-top box by first removing the corners and then by folding the box sides up and securing the tabs to the adjacent box side. The starting cardboard sheet has height ***h*** and width ***w***. When cut and folded, the box has a width of ***w-2x***, a length of ***h-2x***, and a height of ***x***. In order to properly secure the tabs to the adjacent box side, the width of the tab must be 5 centimeters (0.05m). The objective is to **maximize the volume of the box** by choosing an appropriate value of ***x*** (the height of the box) and ***w*** (the starting width of the cardboard sheet).

1. Develop an expression for the volume of the box as a function of ***x*** and ***w*** only. *Hint:* The height ***h*** and width ***w*** are related by the total area.

2. Determine the optimal volume of the box. Differentiate the objective with respect to ***x*** and ***w*** and set each equation equal to zero. Solve the resulting two equations for optimal values of ***x*** and ***w***. Remember that for $ax^2 + bx + c = 0$ the solution is the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

3. Show the first iteration of the steepest descent method starting from *w*=1.0m and *x*=0.1m and alpha=0.2. Do not normalize the search vector. Plot the starting and first iteration point on the contour plot. Remember that (maximize Volume) = (minimize -Volume).



4. What influence do the side tab constraints have on the optimal solution?

Solution Key

1. **V = 0.8 * x – 2*x^2*w – 2 * x^2 * 0.8 / w + 4*x^3**
2. dV/dw = -2*x^2 + 1.6 * x^2 / w^2

   dV/dx = 0.8 – 4*x*w - 3.2 * x / w + 12 * x^2

   Solving the first equation gives:

   -2*x^2 + 1.6 * x^2 / w^2 = 0

   Multiply by w^2 and divide by x^2

   -2 * w^2 + 1.6 = 0

   w^2 = -1.6 / -2.0

   w = sqrt(1.6/2.0) = 0.8944

   Using the second equation to calculate:

   0.8*0.89 – 4*x*0.89^2 - 3.2*x + 12*x^2*0.89 = 0

   0.7155 – 6.4*x + 10.68*x^2 = 0

   x = (-b + sqrt(b^2-4*a*c))/2*a = 0.45 (Infeasible)

   x = (-b - sqrt(b^2-4*a*c))/2*a = 0.1487 (Optimal)
3. Steepest descent

   w = 1.0m x=0.1m objective function f = minimize (-V)

   Gradients

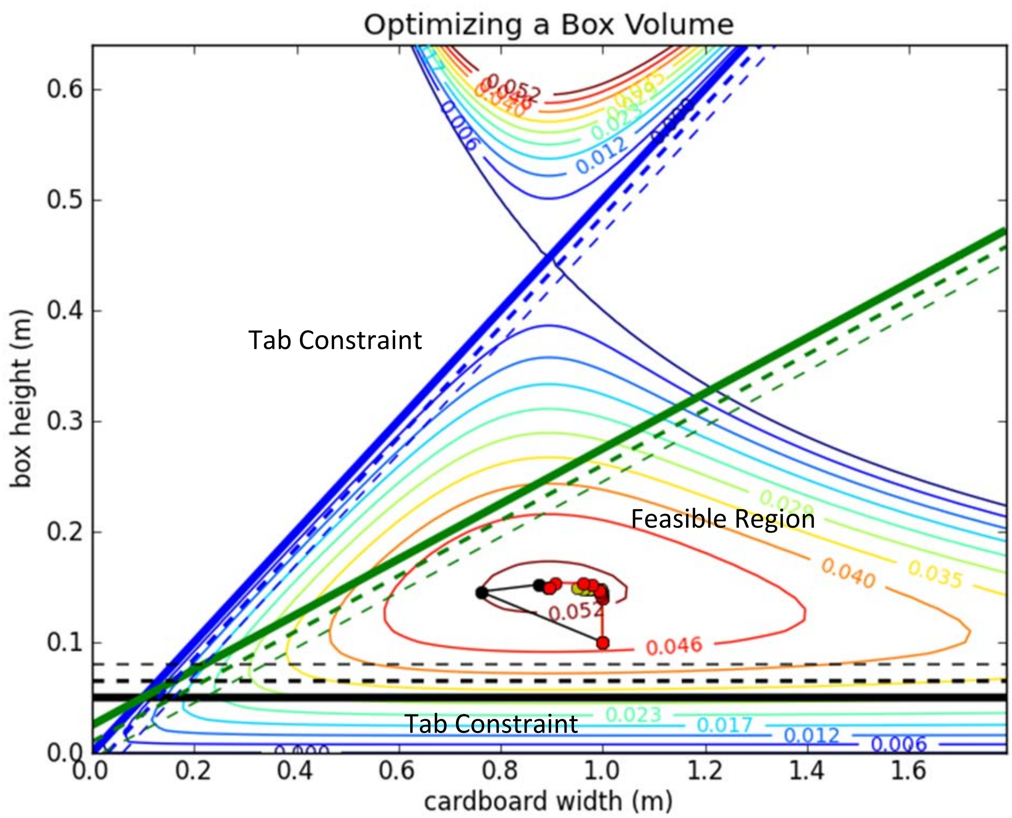
   df/dw = -(-2*x^2 + 1.6 * x^2 / w^2) = 2 * 0.01 - 1.6 * 0.01 / 1 = 0.02-0.016 = +0.004

   df/dx = -(0.8 – 4*x*w - 3.2 * x / w + 12 * x^2) = -0.8 +4*0.1 + 3.2*0.1 - 12 * 0.01

   = -0.8 + 0.4 + 0.32 - 0.12 = -0.2

   x1 **=** x0 + alpha * (-grad(x0)) **=** [1, 0.1] - 0.2 * [0.004, -0.2] **=** [1-0.0008, 0.1+0.04] **= [0.9992, 0.14]**

4. No active constraints at the solution although the width / 2 > x + tab does keep the solution away from a possible second solution that would create a tall box (negative lengths).



Zoomed In to Show Iterations of 4 methods: Newton's, Quasi-Newton (BFGS), CG, and Steepest Descent