Bolt Pattern Design Optimization Project

Sean Johnson, Daniel Koch, Paul Nyholm

15 February 2013

Optimization Problem

Two steel plates of given dimensions are aligned perpendicular to each other, with one corner of each plate touching the origin of the X-Y plane, as shown in Figure 1 below. The two plates will be bolted together using three $\frac{1}{4}$ in. UNC SAE Grade 5 bolts, which will be placed within the boundaries of the square created by the overlap of the two steel plates. A force, P, will then be applied at a location (x_P, y_P) on each plate in the direction shown in Figure 1. It is assumed that the two plates are rigid and do not deflect. It is also assumed that no shear load is carried by friction between the two members. The bolts are subjected to both tension and shear.

The goal of this problem is to find the bolt locations that maximize the force P that can be carried by the bolted joint before it fails. Failure occurs when the shear stresses in any one of the bolts exceeds the yield stress of the bolt. In order simplify the calculations, we will reformulate this problem as an equivalent optimization problem, which is to find the bolt locations that minimize the fraction of the force P that is felt by any one of the bolts.

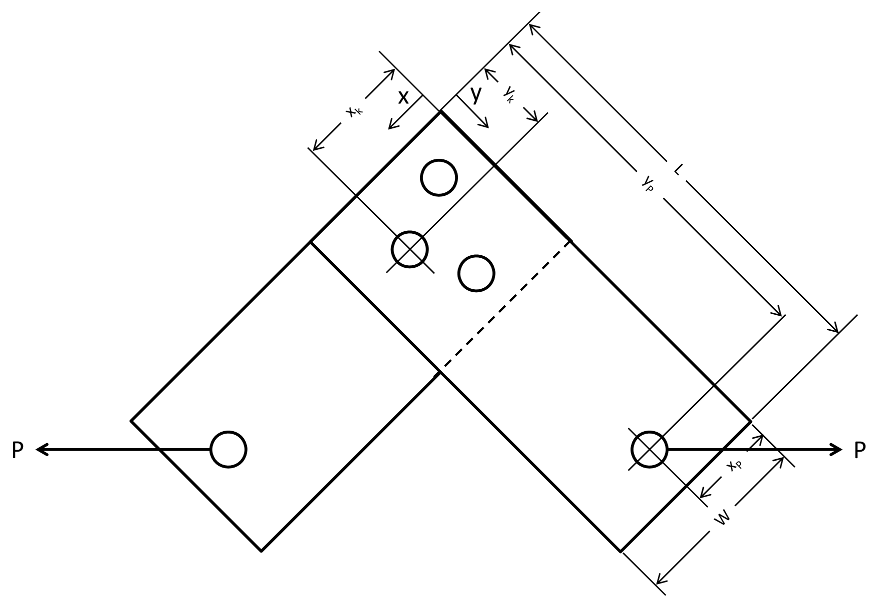

Figure 1: General Setup

Modeling

A nice modeling approach for this problem is to use vectors to represent the various forces acting on each bolt k. These forces include the direct shear force induced from the applied load ($\vec{F}_k'$), the force induced by the moment ($\vec{F}_k''$), and the total force ($\vec{F}_k$) experienced by the bolt, which is the vector sum of the two other forces. The notation used for this problem is described by Figure 2.

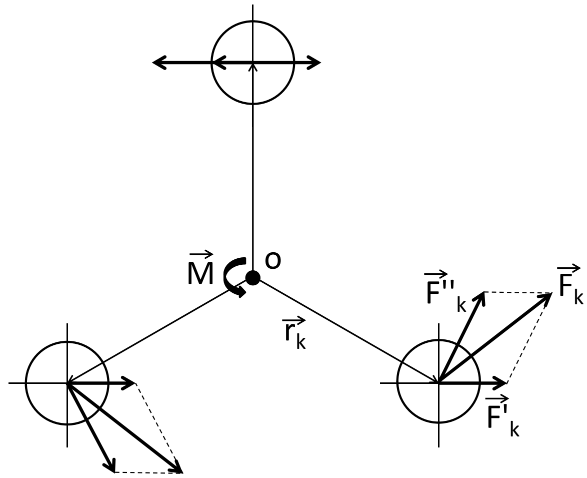

Figure 2: Vector Notation

First, we can model the shear force, from the applied load, on each of the bolts. The applied force vector for the force pulling the plates apart is:

$$\vec{P} = P\left(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

The direct shear load is distributed evenly between the three bolts in the direction of the applied force. This force on each bolt is given by the equation:

$$\begin{aligned}\vec{F}_k' &= \frac{\vec{P}}{3} \\ &= \frac{P}{3}\left(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)\end{aligned}$$

In addition to the shear force on each of the bolts, forces induced by the effective moment between $\vec{P}$ and the centroid of the bolt pattern also exist ($\vec{F}_k''$). In order to find these forces, the moment vector, $\vec{M}$, must first be found. We can calculate $\vec{M}$ using the following cross product:

$$\vec{M} = \vec{R} \times \vec{P}$$

Where $\vec{R}$ is moment arm, or position vector from the centroid of the bolt pattern to the applied load:

$$\vec{R} = (x_P - \bar{x})\,\hat{i} + (y_P - \bar{y})\,\hat{j}$$

Since the cross-sectional areas of all three bolts are the same, the equations needed to find the centroid of the bolt pattern are simplified to:

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}$$

$$\bar{y} = \frac{y_1 + y_2 + y_3}{3}$$

Finally, substituting $\vec{R}$ and $\vec{P}$ we get:

$$\vec{M} = \frac{P}{\sqrt{2}}\left[(x_P - \bar{x}) + (y_P - \bar{y})\right]\hat{k}$$

Now that we have the moment created by the applied load $\vec{P}$, we can calculate how it is distributed among the three bolts. This can be done by first finding the moment arm between each $\vec{F}_k''$ and the centroid of the bolt pattern:

$$\vec{r}_k = (x_k - \bar{x})\,\hat{\imath} + (y_k - \bar{y})\,\hat{\jmath}$$

These moment arms are then crossed with the vector $-\hat{k}$ to find the direction of the moment force for each bolt:

$$\begin{aligned}\hat{f}_k'' &= \frac{1}{\|\vec{r}_k\|}\left(\vec{r}_k \times -\hat{k}\right)\\ &= \frac{1}{\sqrt{(x_k-\bar{x})^2 + (y_k-\bar{y})^2}}\left[-(y_k-\bar{y})\,\hat{\imath} + (x_k-\bar{x})\,\hat{\jmath}\right]\end{aligned}$$

Additionally, we must know the magnitude of each of these forces, which can be found by using the following equation:

$$F_k'' = \frac{\|\vec{M}\| \cdot \|\vec{r}_k\|}{\|\vec{r}_1\|^2 + \|\vec{r}_2\|^2 + \|\vec{r}_3\|^2}$$

Now that we have both the magnitude and direction, we can use their product to find $\vec{F}_k''$:

$$\vec{F}_k'' = F_k''\hat{f}_k$$

With both $\vec{F}_k'$ and $\vec{F}_k''$ known, we can now find the magnitude of the resultant force on each of the bolts:

$$\begin{aligned}\vec{F}_k &= \vec{F}_k' + \vec{F}_k''\\ F_k &= \|\vec{F}_k\|\end{aligned}$$

Note: If we assume P is 1, the resultant force values report the fraction of P that each bolt experiences. Using the minimum proof strength of an SAE Grade 5 bolt, and the minor diameter area, we can calculate the maximum applied force using:

$$P = \frac{S_y A}{F}$$

Constraints

To ensure that the bolt pattern does not fail by tear out (the bolts being too close to the edge of the plate), a safety factor of 1.25 diameters should be used.

$$1.25d \le x_k, y_k \le W - 1.25d$$

Additionally, to ensure that the bolt heads do not interfere with each other, we will add a constraint that the distance between any two bolts must be at least as great as the diameter of the bolt heads:

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \ge d_H \quad \text{for} \quad i, j = 1, 2, 3$$

Data

- $x_P = 1.0\,\text{in}$
- $y_P = 4.75\,\text{in}$
- $P = 1$ (assume unit force for optimization problem)
- Length of steel plates, $L = 5.5\,\text{in}$
- Width of steel plates, $W = 2\,\text{in}$

- Bolt diameter, $d = 0.25\,\text{in}$
- Bolt head diameter, $d_H = 0.5\,\text{in}$
- Minor bolt diameter area, $A = 0.0269\,\text{in}^2$
- Minimum bolt proof strength, $S_y = 85\,\text{kpsi}$

Hints

To maximize the force, you need to minimize the maximum force acting on the bolts (i.e. minimize max(F1,F2,F3)). For example, if bolt three is carrying a greater force than both bolts one and two, then the force on bolt three needs to be minimized. This method treads in some unfamiliar territory for many people, so some sample code is provided below. The code in Listing 1 below shows one way in which you may minimize the maximum force on the bolts. This code should be added to the preamble (prior to the "`Model`" statement) of your APMonitor model file:

```
Objects
    m[1:2] = max
End Objects

Connections
    F1 = m[1].x[1]
    F2 = m[1].x[2]
    m[2].x[1] =m[1].y
    F3 = m[2].x[2]
    max_F = m[2].y
End Connections
```

Listing 1: Code for calculating the maximum of F_1, F_2, and F_3.

This code compares the forces on bolts one and two and outputs the larger of the two forces. That resulting force is then compared to the force on bolt three and the largest of those two forces is the final output. It effectively chooses the largest of the forces on the three bolts and defines it as `max_F`. (Note that `F1`, `F2`, and `F3` must be defined as variables in your model and calculated appropriately) You can then minimize this force to obtain the optimal locations for the bolts using a simple objective function. In the "Equations" portion of your model file, include something like the objective function shown in Listing 2:

```
minimize max_F
```

Listing 2: Example objective function for minimizing the maximum of the three bolt forces.

Using this as our objective will ensure that all of the bolts will carry the load as equally as possible. Ideally, the three bolts will all fail at the same time, which means all of the potential strength of the bolts is being utilized.