Mathcad Lecture #6 In-class Worksheet
Solving Equations and Optimizing Functions

At the end of this lecture, you will be able to:

- solve for the roots of a polynomial using `polyroots`.
- obtain approximate solutions to single equations from `tracing` a graph.
- obtain a solution to a single non-linear equation using the `root` function.
- solve systems of non-linear equations using "`Solve Blocks`"

1. Solving for the roots of a polynomial.

Description

The polyroots() function is simple and powerful. It finds ALL the roots of a polynomial, both real and imaginary.

Demonstration

Find all the roots, both real and imaginary, of the following equation.

$$x^6 - 2 \cdot x^5 - 3 \cdot x^4 + 3 \cdot x^3 - x^2 + 2 \cdot x = 0$$

Step 1: Define the input matrix containing the coefficients of each term.

$$coeff := \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \\ -3 \\ -2 \\ 1 \end{pmatrix}$$

Key Points:
1. The coefficient matrix is a single column with n+1 rows where n is the order of the polynomial.
2. The coefficient matrix is ordered from x^0 to x^n.
3. Polynomial must be place in form where RHS = 0.

Step 2: Use the polyroots function.

$$polyroots(coeff) = \begin{pmatrix} -1.599 \\ -0.056 - 0.677i \\ -0.056 + 0.677i \\ 0 \\ 1 \\ 2.712 \end{pmatrix}$$

Practice

Solve for all the roots, both real and imaginary, of the following equation.

$$x^4 + x = 3$$

$$coeff_2 := \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 4 \end{pmatrix}$$

$$polyroots(coeff_2) = \begin{pmatrix} -1 \\ 0.072 + 0.933i \\ 0.072 - 0.933i \\ 0.856 \end{pmatrix}$$

2. Obtaining Approximate Solutions to Single Equations by "Tracing".

Background

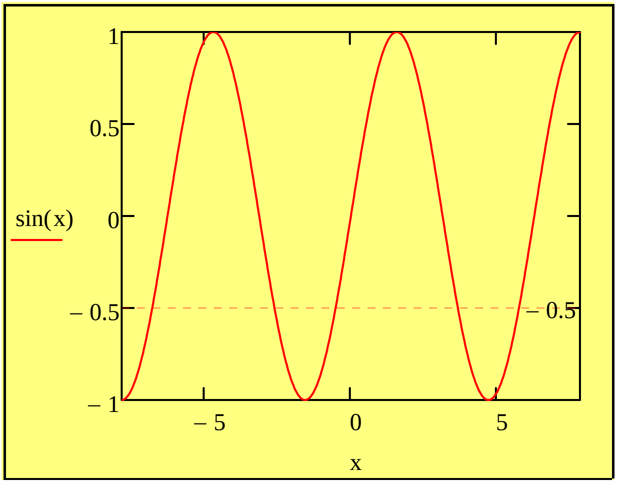
Remember, roots of a single equation can be found from a graph. If the equation is of the form $LHS = 0$, then the roots are found where the graph crosses the x-axis. If the equation is of the form $LHS = constant$, the roots are found where the graph crosses the $y = constant$ line.

Demonstration

Find approximate values for all the roots of the following equation from -2.5π to 2.5π.

$$\sin(x) = -0.5$$

Step 1: Graph the equation and place a line at y = -0.5.
(Ensure the ranges are correct.)



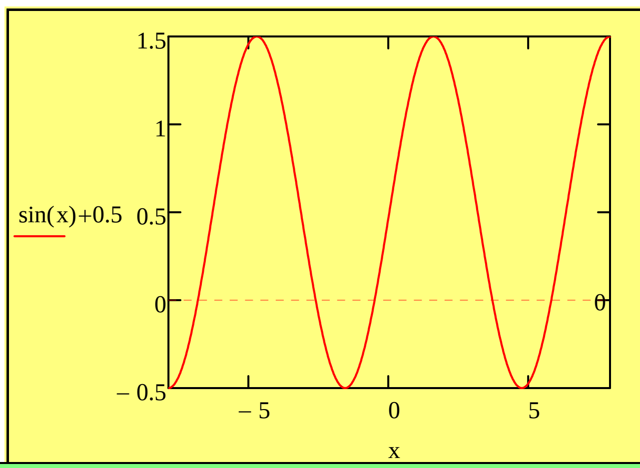
Approximate Roots
-6.8
-2.6
-0.53
3.8

Step 2: Trace the plot.

A. You can add horizontal or vertical lines to a plot, at specific values of y and x respectively, by placing a "marker" on the graph. This is done by:
- Double clicking on the graph
- Checking "Show markers" on the axis desired and clicking OK
- filling in one of the place holders that appeared with a numerical value

B. The Trace utility, on the Graph tool palette, is useful to determine approximate values for roots.
- Click on the graph
- Open the Graph tool palette from the Math tool palette
- Click on the trace button
- You can then move the tracking point by the arrow keys or the mouse.

Alternative Solution



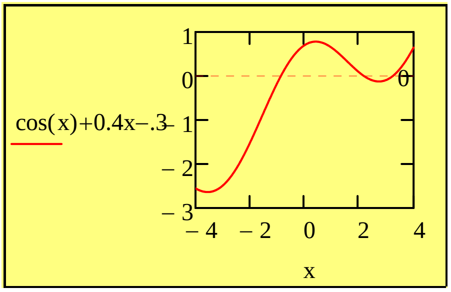
Approximate Roots
-6.8
-2.6
-0.53
3.7

Key Point: Graphing is often done to get *initial guesses* for the iterative solvers described in the next sections.

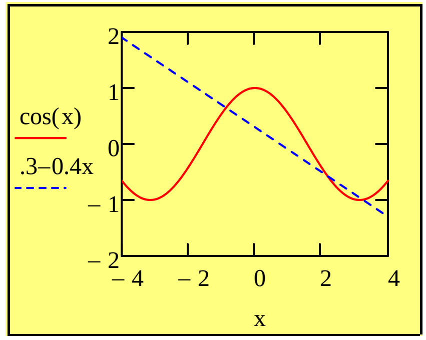
Practice

Find approximate value for the all the roots of the following equation.

$$\cos(x) = 0.3 - 0.4x$$



Approximate Roots
-0.88
2.2
3.2



3. Solving Single Equations Using the root() Function.

Description

The root() function offers a method of finding the solution to single equations of any type, linear or non-linear.

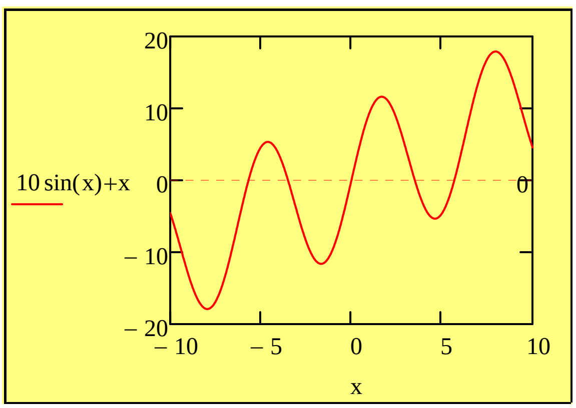
Demonstration

Find the roots of the following equation: $10\sin(x) = -x$

Step 1: Put the equation in the form $f(x) = 0$.

$$10\sin(x) + x = 0$$

Step 2: The root function requires a guess value so first plot the function.



Step 3: Use root function with different guesses to find all the roots.

Guesses	Solutions
$a := -6$	$\text{root}(10\cdot\sin(a) + a, a) = -5.679$
$a := -3$	$\text{root}(10\cdot\sin(a) + a, a) = -3.499$
$a := 0$	$\text{root}(10\cdot\sin(a) + a, a) = 0$
$a := 3$	$\text{root}(10\cdot\sin(a) + a, a) = 3.499$
$a := 6$	$\text{root}(10\cdot\sin(a) + a, a) = 5.679$

Alternate Method: Define a function.

$$f(x) := 10\cdot\sin(x) + x$$

Guess	Solution
$b := -6$	$\text{root}(f(b), b) = -5.679$

Key Points:
1. The root function needs a guess value. You can get the guess value from a graph of the function.
2. The guess value is the second argument of function. The first argument is the function written in terms of the guess value.

Demo

The van der Waals equation of state, $P = \frac{R \cdot T}{v - b} - \frac{a}{v^2}$ describes the PVT behavior of real gases better than the ideal gas equation of state. For butane, a = 1.3701x10^7 atm cm^6 mol^{-2} and b = 116.4 cm^3 mol^{-1}. Using the van der Waals EOS, calculate the liquid and vapor volume of butane at 100 °C and 15.41 bar.

Step 1: Define a function of the form f(x)=0.

$R_g := 8.314 \frac{J}{mol \cdot K}$ $\quad a := 1.3701 \cdot 10^7 \frac{atm \cdot cm^6}{mol^2}$ $\quad b := 116.4 \frac{cm^3}{mol}$

$t := (100 + 273.15) \cdot K$ $\quad p := 15.41 bar$

$pvt_{vdw}(v) := \frac{R_g \cdot t}{v - b} - \frac{a}{v^2} - p$

Step 2: Use the root function to find the volumes. Remember, a good guess for the vapor volume is RT/P and a good guess for the liquid volume is 1.1b.

Guesses	Solutions	
$v_g := \frac{R_g \cdot t}{p}$	$root(pvt_{vdw}(v_g), v_g) = 1.611 \times 10^3 \cdot \frac{cm^3}{mol}$	Vapor Volume
$v_g := 1.1 \cdot b$	$root(pvt_{vdw}(v_g), v_g) = 212.465 \cdot \frac{cm^3}{mol}$	Liquid Volume

4. Solving Systems of Nonlinear Equations Using Solve Blocks (Given/Find Blocks)

Description

Solve blocks (given/find blocks) can be used to solve systems of non-linear equations. If you have a system of linear equations, matrix math should be used to obtain the solutions. If the equations are not linear, the only other way to solve them in Mathcad is using a solve block.

The Procedure

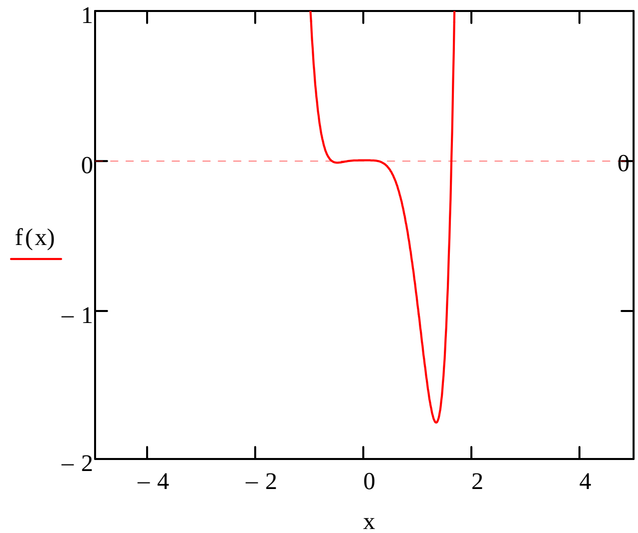
1. Define the **guess** value for each unknown.
2. Initiate a Solve Block with the key work **Given**
3. Enter the **equations** to be solved.
   a. You must use a bold equals to define the equations.
   b. You must also use the variable names used for the guess values.
4. Complete the solve block with a **find** statement.
   a. find(x,y,....)
   b. The arguments to the find statement are the unknowns.

Demonstration for Single Equation

Use a Solve Block to find the largest solution of the equation: $x^6 - x^5 - x^4 = 0$

Step 0: Plot the function to get guesses.

$f(x) := x^6 - x^5 - x^4$



Step 1: Guess $x_g := 2$

Step 2: Given Given

Step 3: Equation $x_g^6 - x_g^5 - x_g^4 = 0$

Step 4: Find $ans := \text{Find}(x_g)$

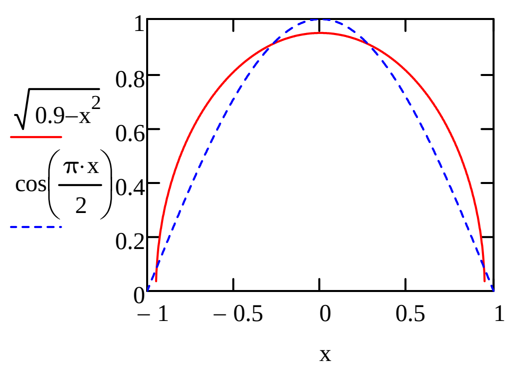
$ans = 1.618$

Key Points:
1. Use := Outside of the given block and bold = inside the given block.
2. The equations inside the given block must be written in terms of the guess variable.

Demonstration for Multiple Equations

Find a solution that satisfies the following equations: $x^2 + y^2 = 0.9$ $y = \cos\left(\frac{\pi x}{2}\right)$

Step 0: Plot the functions to get guesses.



Step 1: Guesses $x_g := -.5$ $y_g := .9$

Step 2: Given Given

Step 3: Equations $x_g^2 + y_g^2 = 0.9$

$y_g = \cos\left(\frac{\pi \cdot x_g}{2}\right)$

Step 4: Find $\begin{pmatrix} xx \\ yy \end{pmatrix} := \text{Find}(x_g, y_g)$ $xx = -0.276$ $yy = 0.908$

Key Points:
1. You need a guess value for each unknown.
2. Any variable not placed as an argument inside the find() is considered a constant.

Other Solve Block Tips

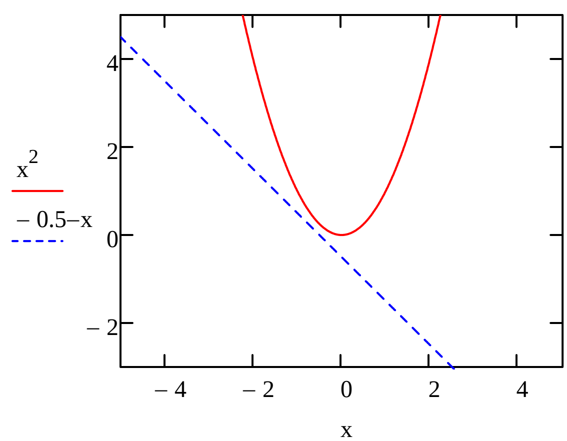
1. The fewer number of equations inside the given block, the easier it is for the solution to converge.
2. You can include Boolean operators (<, >, etc.) inside the given block to create constraints.
3. If you want to obtain other solutions to the same system of equations, you can copy/paste the entire solve block to a new location on the sheet and just change the guesses.
4. A solve block can also be terminate with the minerr(), maximize() and minimize() statements.

Minerr Demonstration

Background: Sometimes, the solve block cannot converge to a solution, but you want the best solution to equation. Minerr can be used to find the values of the unknowns that minimizes the error.

Find the x and y that best satisfy the following expressions. $y = x^2$ $y = -0.5 - x$

Step 0: Plot the functions to get guesses.



Step 1: Guesses $x_g := -1$ $y_g := 1$

Step 2: Given Given

Step 3: Equations $y_g = x_g^2$

$y_g = -0.5 - x_g$

Step 4: Find $ans := \text{Minerr}(x_g, y_g)$ $ans = \begin{pmatrix} -0.5 \\ 0.125 \end{pmatrix}$

Extra Practice 1

Find a solution to the following system of equations. $x^2 + 10 \cdot y = (4 \cdot x^2 - 2 \cdot \ln(y)) \cdot \sqrt{e^{x \cdot y}}$ $4 \cdot x + 3 \cdot x \cdot y = 2 \cdot \frac{y}{x}$

$x_g := 1$ $y_g := 1$

Given

$x_g^2 + 10 \cdot y_g = (4 \cdot x_g^2 - 2 \cdot \ln(y_g)) \cdot \sqrt{e^{x_g \cdot y_g}}$ $4 \cdot x_g + 3 \cdot x_g \cdot y_g = 2 \cdot \frac{y_g}{x_g}$

$\begin{pmatrix} x \\ y \end{pmatrix} := \text{Find}(x_g, y_g)$ $x = 0.348$ $y = 0.296$

Extra Practice 2

In piping systems, friction causes the pressure of the liquid to drop as it flow through the pipe. The friction factor, f, is a measure of the amount of pressure loss due to friction. It can be found from the following relationship:

$$\frac{1}{\sqrt{\frac{f}{2}}} = 2.5 \cdot \ln\left(Re \cdot \sqrt{\frac{f}{8}}\right) + 1.75$$

where Re is the Reynolds number, a measure of the relative importance of the inertial and viscous forces. For Re = 25,000, determine the friction factor.

$f_g := .001$ $Re := 25000$

Given

$$\frac{1}{\sqrt{\frac{f_g}{2}}} = 2.5 \cdot \ln\left(Re \cdot \sqrt{\frac{f_g}{8}}\right) + 1.75$$

$f := Find(f_g)$ $f = 6.108 \times 10^{-3}$

Extra Practice 3

The van der Waals equation of state, $P = \frac{R \cdot T}{v - b} - \frac{a}{v^2}$ describes the PVT behavior of real gases better than the ideal gas equation of state. For butane, a = 1.3701x10^7 atm cm^6 mol^{-2} K^{-1} and b = 116.4 cm^3 mol^{-1}. Using the van der Waals EOS, calculate the liquid and vapor volume of butane at 100 °C and 15.41 bar.

$a = 1.388 \frac{m^5 \cdot kg}{mol^2 \cdot s^2}$ $b = 1.164 \times 10^{-4} \frac{m^3}{mol}$ $R_g = 8.314 \frac{m^2 \cdot kg}{mol \cdot K \cdot s^2}$

$p = 1.541 \times 10^6 \, Pa$ $t = 373.15 \, K$

$v_g := \frac{R_g \cdot t}{p}$ Given $p = \frac{R_g \cdot t}{v_g - b} - \frac{a}{v_g^2}$ $v_{gas} := Find(v_g)$ $v_{gas} = 1.611 \times 10^3 \cdot \frac{cm^3}{mol}$

$v_g := 1.1 \cdot b$ Given $p = \frac{R_g \cdot t}{v_g - b} - \frac{a}{v_g^2}$ $v_{liq} := Find(v_g)$ $v_{liq} = 212.465 \cdot \frac{cm^3}{mol}$